

THE GOTO NUMBERS OF PARAMETER IDEALS

WILLIAM HEINZER AND IRENA SWANSON

ABSTRACT. Let Q be a parameter ideal of a Noetherian local ring (R, \mathbf{m}) . The Goto number $g(Q)$ of Q is the largest integer g such that $Q : \mathbf{m}^g$ is integral over Q . We examine the values of $g(Q)$ as Q varies over the parameter ideals of R . We concentrate mainly on the case where $\dim R = 1$, and many of our results concern parameter ideals of a numerical semigroup ring.

1. INTRODUCTION

This note started from the group work at the workshop “Integral closure, multiplier ideals, and cores” that took place at the American Institute of Mathematics (AIM) in Palo Alto, California, in December 2006. Shiro Goto presented the background, motivation, and some intriguing open questions.

Recall that if (R, \mathbf{m}) is a Noetherian local ring with $\dim R = d$, then an \mathbf{m} -primary ideal Q is called a *parameter ideal* if Q is generated by d elements.

A motivating result for the group work at AIM is:

Theorem 1.1. (Corso, Huneke, Vasconcelos [2], Corso, Polini [4], Corso, Polini, Vasconcelos [5], Goto [6]) *Let (R, \mathbf{m}) be a Cohen-Macaulay local ring of positive dimension. Let Q be a parameter ideal in R and let $I = Q : \mathbf{m}$. Then the following are equivalent:*

- (1) $I^2 \neq QI$.
- (2) *The integral closure of Q is Q .*
- (3) *R is a regular local ring and $\mu(\mathbf{m}/Q) \leq 1$.*

Consequently, if (R, \mathbf{m}) is a Cohen-Macaulay local ring that is not regular, then $I^2 = QI$. If $\dim R > 1$, it follows that the Rees algebra $R[It]$ is a Cohen-Macaulay ring, and even without the assumption that $\dim R > 1$, the fact that $I^2 = QI$ implies that the associated graded ring $\operatorname{gr}_I(R) = R[It]/IR[It]$ and the fiber ring $R[It]/\mathbf{m}R[It]$ are both Cohen-Macaulay.

In [7], Goto, Matsuoka, and Takahashi explore the Cohen-Macaulayness and Buchsbaumness of the associated graded and fiber rings and Rees algebras for

ideals $I = Q : \mathbf{m}^2$ under the condition that $I^3 = QI^2$. They also give examples showing that Cohen-Macaulayness does not always hold. Notice that the condition $I^3 = QI^2$ implies that I is integral over Q , so $I \subseteq \overline{Q}$, where \overline{Q} denotes the integral closure of Q [13, Corollary 1.1.8].

It seems that a natural next step would be to explore the Cohen-Macaulay property for the various ring constructs from the ideal $I = Q : \mathbf{m}^3$. We expect the necessity of even further restrictions on R and I . However, rather than examining each of $I = Q : \mathbf{m}^i$ for increasing i in turn, we pass to examining $I = Q : \mathbf{m}^g$, where g is the greatest integer such that $Q : \mathbf{m}^g$ is integral over Q . Because of the pioneering work Shiro Goto has done in this area we define the Goto number of a parameter ideal Q as follows:

Definition 1.2. Let Q be a parameter ideal of the Noetherian local ring (R, \mathbf{m}) . The largest integer g such that $Q : \mathbf{m}^g$ is integral over Q is denoted $g(Q)$ and called the *Goto number* of Q . In the case where $\dim R = 1$ and $Q = xR$, we sometimes write $g(x)$ instead of $g(Q)$.

Notice that the Goto number $g(Q)$ is well defined, for $Q : \mathbf{m}^0 = Q : R = Q$ is integral over Q , and for sufficiently large n , $\mathbf{m}^n \subseteq Q$, so $Q : \mathbf{m}^n = R$, which is not integral over Q .

During the workshop we concentrated on various invariants, dubbed “Goto invariants of a Noetherian local ring (R, \mathbf{m}) ”, that involve the Goto numbers of parameter ideals. These invariants are discussed in Section 2. During our subsequent work, we decided that the set

$$\mathcal{G}(R) = \{g(Q) \mid Q \text{ is a parameter ideal of } R\},$$

where R is a fixed one-dimensional Noetherian local ring is a possibly more interesting invariant. Most of the paper has to do with an examination of the integers that are in $\mathcal{G}(R)$. In the case where (R, \mathbf{m}) is an arbitrary one-dimensional Noetherian local ring, we prove the existence of a positive integer n such that every parameter ideal contained in \mathbf{m}^n has Goto number the minimal integer in $\mathcal{G}(R)$. With additional hypothesis on R , we prove that the set $\mathcal{G}(R)$ is finite.

Our notation is mainly as in [13]. In particular, we use \overline{R} to denote the integral closure of the ring R , and \overline{J} to denote the integral closure of the ideal J of R . For many of the examples in the paper, the calculations were done using the symbolic computer algebra system Macaulay2 [8].

For much of the paper we focus on a special type of one-dimensional Noetherian local domain. As in the monograph of Jürgen Herzog and Ernst Kunz [10], we consider a rank-one discrete valuation domain V with field of fractions K and let $v : K \setminus \{0\} \rightarrow \mathbb{Z}$ denote the normalized valuation associated to V . Thus if $x \in V$ generates the maximal ideal of V , then $v(x) = 1$. Associated with each subring R of V is a subsemigroup $G(R) = \{v(r) \mid r \in R \setminus \{0\}\}$ of the additive semigroup \mathbb{N}_0 of nonnegative integers. $G(R)$ is the *value semigroup* of R with respect to V .

Definition 1.3. A subring R of V is called a *numerical semigroup ring associated to V* if it satisfies the following properties:

- (1) R has field of fractions K and the integral closure of R is V .
- (2) V is a finitely generated R -module.
- (3) There exists $x \in V$ with $v(x) = 1$ such that $x^n \in R$ for each integer n such that $n = v(r)$ for some $r \in R$, and if $\mathfrak{m} = xV \cap R$, then the canonical injection $R/\mathfrak{m} \hookrightarrow V/xV$ is an isomorphism.

The value semigroup $G(R)$ is the *numerical semigroup associated to R*

Remark 1.4. Let R be a numerical semigroup ring associated to the valuation domain V as in (1.3). We then have the following.

- (1) Since V is a finitely generated R -module, R is a one-dimensional Noetherian local domain with maximal ideal \mathfrak{m} [11, Theorem 3.7].
- (2) Since the conductor [13, page 234] of R in V is nonzero, the value semigroup $G(R) = \{v(r) \mid r \in R \setminus \{0\}\}$ contains all sufficiently large integers. The largest integer f that is not in $G(R)$ is called the *Frobenius number* of R , and $C := x^{f+1}V$ is the conductor of R in V .
- (3) If $0 < a_1 < a_2 < \dots < a_d$ are elements of $G(R)$ that generate $G(R)$, then $\mathfrak{m} = (x^{a_1}, x^{a_2}, \dots, x^{a_d})R$.
- (4) An application of Nakayama's lemma [11, Theorem 2.2] implies $R[x] = V$.
- (5) If u is a unit of V , then $R/\mathfrak{m} = V/xV$ implies there exists a unit u_0 of R such that $u - u_0 \in xV$. If $u \neq u_0$, there exists a positive integer i such that $u - u_0 = wx^i$, where w is a unit of V . Repeating the above process on w , we see that every unit u of V has the form

$$u = u_0 + u_1x + \dots + u_fx^f + \alpha,$$

where $\alpha \in C$, u_0 is a unit of R , and each u_i , $1 \leq i \leq f$, is either zero or a unit of R .

- (6) Every nonzero element $r \in R$ has the form $r = ux^b$ for some $b \in G$ and some unit $u \in V$. Multiplying u by a unit in R and using item (5), we see that every nonzero principal ideal of R has the form ux^bR , where

$$u = 1 + u_1x + u_2x^2 + \cdots + u_fx^f + \alpha,$$

where $\alpha \in C$ and each u_i is either zero or a unit of R . Thus

$$ux^b = (1 + u_1x + u_2x^2 + \cdots + u_fx^f)x^b + \alpha x^b.$$

Since $ux^b \in R$, it follows that $b + i \in G$ for each i such that $u_i \neq 0$. Also $\alpha \in C$ implies $\alpha = u\beta$, where $\beta \in C$. Thus

$$ux^b - \alpha x^b = ux^b - u\beta x^b = ux^b(1 - \beta).$$

Since $1 - \beta$ is a unit of R , we conclude that each nonzero principal ideal of R has the form $(1 + u_1x + \cdots + u_fx^f)x^bR$, where $b \in G$, each u_i is either zero or a unit of R , and if $u_i \neq 0$, then $b + i \in G$.

- (7) With $r = ux^b$, if we pass to integral closure, we have

$$\overline{(r)} = \overline{(r)}\overline{V} \cap R = \overline{(x^b)}\overline{V} \cap R = (x^e : e \in G, e \geq b)R.$$

Remark 1.5. With additional assumptions about the rank-one discrete valuation domain V it is possible to realize numerical semigroup rings by starting with the group. Let k be a field and let x be an indeterminate over k . If V is either the formal power series ring $k[[x]]$ or the localization of the polynomial ring $k[x]$ at the maximal ideal generated by x , then for each subsemigroup G of \mathbb{N}_0 that contains all sufficiently large positive integers, there exists a numerical semigroup ring R associated to V such that $G(R) = G$. In each case one takes generators a_1, \dots, a_d for G . If V is the formal power series ring $k[[x]]$, then $R = k[[x^{a_1}, \dots, x^{a_d}]]$ is the subring of $k[[x]]$ generated by all power series in x^{a_1}, \dots, x^{a_d} , while if V is $k[x]$ localized at the maximal ideal generated by x , then R is $k[x^{a_1}, \dots, x^{a_d}]$ localized at the maximal ideal generated by x^{a_1}, \dots, x^{a_d} .

We observe in Proposition 1.6 a useful result for computing Goto numbers of parameter ideals in dimension one.

Proposition 1.6. *Let Q_1 and Q_2 be ideals of a Noetherian local ring (R, \mathfrak{m}) . Assume that Q_2 is not contained in any minimal prime of R . If e is a positive integer such that $Q_1 : \mathfrak{m}^e$ is not integral over Q_1 , then $Q_1Q_2 : \mathfrak{m}^e$ is not integral over Q_1Q_2 .*

Proof. It suffices to check integral closure modulo each minimal prime ideal, so we may assume that R is an integral domain [13, Proposition 1.1.5]. Let $x \in Q_1 : \mathfrak{m}^e$. Then $xQ_2 \subseteq Q_1Q_2 : \mathfrak{m}^e$. If all the elements in xQ_2 are integral over Q_1Q_2 , then [13, Corollary 6.8.7] implies that x is integral over Q_1 . \square

In dimension one, the product of two parameter ideals is again a parameter ideal. Thus Proposition 1.6 has the following immediate corollary.

Corollary 1.7. *Let (R, \mathfrak{m}) be a one-dimensional Noetherian local ring. If Q_1 and Q_2 are parameter ideals of R , then $g(Q_1Q_2) \leq \min\{g(Q_1), g(Q_2)\}$.*

A strict inequality may hold in Corollary 1.7 as we illustrate in Example 1.8.

Example 1.8. Let $G = \langle 3, 5 \rangle$ be the numerical subsemigroup of \mathbb{N}_0 generated by 3 and 5, and let R as in Remark 1.5 be a numerical semigroup ring such that $G(R) = \langle 3, 5 \rangle$. A direct computation shows that the parameter ideal $Q = x^5R$ has Goto number $g(x^5) = 3$, while $Q^2 = x^{10}R$ has the property that $x^9 \in x^{10}R : \mathfrak{m}^3$. Therefore $x^{10}R : \mathfrak{m}^3$ is not integral over $x^{10}R$ and $g(x^{10}) = 2$.

The Goto numbers of parameter ideals of a Gorenstein local ring may be described using duality as in Proposition 1.9.

Proposition 1.9. *Let Q be a parameter ideal of a Gorenstein local ring (R, \mathfrak{m}) . Assume that $Q \subsetneq \overline{Q}$. Let $J = Q : \overline{Q}$. Then*

$$g(Q) = \max\{i \mid J \subseteq \mathfrak{m}^i + Q\}.$$

Proof. Since R/Q is a zero-dimensional Gorenstein local ring, $(Q : J) = \overline{Q}$, and $(Q : \mathfrak{m}^i) \subseteq \overline{Q}$ if and only if $J \subseteq \mathfrak{m}^i + Q$, cf. [1, (3.2.12)]. \square

2. GOTO INVARIANTS OF LOCAL RINGS NEED NOT BE BOUNDED

Since a regular local ring of dimension one is a rank-one discrete valuation domain, the Goto number of every parameter ideal is 0 in this case. We prove below that in a two-dimensional regular local ring, the Goto number of a parameter ideal Q is precisely $\text{ord } Q - 1$, where $\text{ord } Q$ is the highest power of \mathfrak{m} that contains Q . Thus in a two-dimensional regular local ring, the Goto number of a parameter ideal is uniquely determined by the order of the parameter ideal. It seems natural to expect at least for many local rings (R, \mathfrak{m}) that the Goto number $g(Q)$ becomes

larger as Q is in higher and higher powers of \mathbf{m} . The following are several invariants of a local ring (R, \mathbf{m}) involving Goto numbers $g(Q)$ of parameter ideals Q of R .

$$\begin{aligned} \text{goto}_1(R) &= \sup \left\{ \frac{g(Q)}{\text{ord}(Q)} \mid Q \text{ varies over parameter ideals of } R \right\}, \\ \text{goto}_2(R) &= \sup \left\{ \frac{g(Q)}{\text{ord}(Q : \mathbf{m})} \mid Q \text{ varies over parameter ideals of } R \right\}, \\ \text{goto}_3(R) &= \sup \left\{ \frac{g(Q)}{\text{ord}(Q : \mathbf{m}^{g(Q)})} \mid Q \text{ varies over parameter ideals of } R \right\}. \end{aligned}$$

In order to avoid division by zero, in the definition of $\text{goto}_2(R)$, we exclude the case where R is a regular local ring and $Q = \mathbf{m}$.

Example 2.1 demonstrates the existence, for every integer $d \geq 3$, of a regular local ring (R, \mathbf{m}) of dimension d for which each of the invariants $\text{goto}_i(R)$, $i \in \{1, 2, 3\}$, is infinity.

Example 2.1. Let k be a field, d an integer > 2 , x_1, \dots, x_d variables over k . Let $n \geq e$ be positive integers, and let $Q = (x_1^e, x_2^n, \dots, x_d^n)$. Then $g(Q) = (d-2)(n-1) + e - 1$. For we have:

$$(x_1^e, x_2^n, \dots, x_d^n) : (x_1, \dots, x_d)^{(d-2)(n-1)+e-1} = (x_1^e) + (x_1, \dots, x_d)^n,$$

which is integral over Q , and

$$(x_1^e, x_2^n, \dots, x_d^n) : (x_1, \dots, x_d)^{(d-2)(n-1)+e}$$

contains x_2^{n-1} , which is not integral over Q . Furthermore,

$$\text{ord}(Q) = \text{ord}(Q : \mathbf{m}) = \text{ord}(Q : \mathbf{m}^{(d-2)(n-1)+e-1}) = e.$$

Thus, for each $i \in \{1, 2, 3\}$, $\text{goto}_i(R) \geq \frac{(d-2)(n-1)+e-1}{e}$ for all $n \geq e$. Since $d > 2$, we have $\text{goto}_i(R) = \infty$.

In the case where (R, \mathbf{m}) is a two-dimensional regular local ring, we prove in Theorem 2.2 that the Goto number of a parameter ideal Q depends only on the order of Q .

Theorem 2.2. *Let (R, \mathbf{m}) be a two-dimensional regular local ring. Then for each parameter ideal Q of R , the Goto number $g(Q) = \text{ord}(Q) - 1$.*

Proof. Passing to the faithfully flat extension $R[X]_{\mathbf{m}R[X]}$ preserves the parameter ideal property and its order and Goto number, so that without loss of generality we may assume that R has an infinite residue field. Let $k = \text{ord } Q$. The proof of

[14, Theorem 3.2] shows that $k - 1 \leq g(Q)$. (In Wang's notation in that proof, it is shown that $Q : \mathbf{m}^{k-1} \subseteq (Q \mathbf{m}^{k-1} : \mathbf{m}^{k-1}) \subseteq \overline{Q}$.) Now we prove that $g(Q) \leq k - 1$. Let $Q = (a, b)$. Let $x \in \mathbf{m} \setminus \mathbf{m}^2$ be such that $\text{ord}(a) = \text{ord}(a(R/(x)))$ and $\text{ord}(b) = \text{ord}(b(R/(x)))$. Since the residue field of R is infinite, it is possible to find such an element x . The condition needed for x is that its image in the associated graded ring $\text{gr}_{\mathbf{m}}(R)$ is not a factor of the images of a and b in $\text{gr}_{\mathbf{m}}(R)$. Since $R/(x)$ is a one-dimensional regular local ring, hence a principal ideal domain, by possibly permuting a and b we may assume that $b \in (a, x)$. By subtracting a multiple of a from b , without loss of generality $b = b_0x$ for some $b_0 \in R$. Note that $(a, x) = \mathbf{m}^k + (x)$, and $\text{ord } a = \text{ord } Q = k \leq \text{ord } b$. It follows that $b_0 \mathbf{m}^k \subseteq b_0(a, x) \subseteq (a, xb_0) \subseteq (a, b)$. However, $b_0 \notin \overline{(a, b)}$: otherwise for all discrete valuations v centered on \mathbf{m} , $v(b_0) \geq \min\{v(a), v(b)\}$, whence since $v(b) = v(b_0x) > v(b_0)$, necessarily $v(b_0) \geq v(a)$ for all such v , so that $b_0 \in \overline{(a)} = (a)$, contradicting the assumption that (a, b) is a parameter ideal. This proves that $g(Q) < k = \text{ord } Q$. \square

If (R, \mathbf{m}) is a regular local ring, then the powers of \mathbf{m} are integrally closed. Hence, in this case, if $Q : \mathbf{m}^i$ is integral over Q , then $\text{ord } Q = \text{ord}(Q : \mathbf{m}^i)$. Thus Theorem 2.2 implies the following:

Corollary 2.3. *If (R, \mathbf{m}) is a two-dimensional regular local ring, then each of the invariants $\text{goto}_i(R)$, $i \in \{1, 2, 3\}$, is one.*

Remark 2.4. Let $(\widehat{R}, \widehat{\mathbf{m}})$ denote the \mathbf{m} -adic completion of the Noetherian local ring (R, \mathbf{m}) . Since $R/I \cong \widehat{R}/I\widehat{R}$ for each \mathbf{m} -primary ideal I of R , the \mathbf{m} -primary ideals of R are in one-to-one inclusion preserving correspondence with the $\widehat{\mathbf{m}}$ -primary ideals of \widehat{R} . Also, if I is an \mathbf{m} -primary ideal, then $\overline{I\widehat{R}}$ is the integral closure of $I\widehat{R}$ [13, Lemma 9.1.1]. Since $R/\mathbf{m} \cong \widehat{R}/\widehat{\mathbf{m}}$, and since each parameter ideal of \widehat{R} has the form $Q\widehat{R}$, where Q is a parameter ideal of R , the set $\{\ell_R(\overline{Q}/Q) \mid Q \text{ is a parameter ideal of } R\}$ is identical to the corresponding set for \widehat{R} . Since \widehat{R} is flat over R , we also have $(Q :_R \mathbf{m}^i)\widehat{R} = (Q\widehat{R} :_{\widehat{R}} \widehat{\mathbf{m}}^i)$ for each positive integer i . Therefore, for each parameter ideal Q of R , the Goto number $g(Q) = g(Q\widehat{R})$. Hence the set $\mathcal{G}(R) = \{g(Q) \mid Q \text{ is a parameter ideal of } R\}$ is identical to the corresponding set $\mathcal{G}(\widehat{R})$ for \widehat{R} .

3. ONE-DIMENSIONAL NOETHERIAN LOCAL RINGS

Throughout this section, let (R, \mathbf{m}) be a one-dimensional Noetherian local ring. In subsequent sections we restrict to the special case where R is a numerical semi-group ring. If R is a regular local ring, then it is a principal ideal domain, and hence the Goto number $g(Q) = 0$ for every parameter ideal Q . Thus to get more interesting variations on the Goto number of parameter ideals, we restrict our attention to non-regular one-dimensional Noetherian local rings.

Corollary 1.7 is useful for examining the Goto number of parameter ideals. We observe in Theorem 3.1 that the Goto number of parameter ideals in a sufficiently high power of the maximal ideal of R are all the same and that this eventually constant value is the minimal possible Goto number of a parameter ideal of R .

Theorem 3.1. *Let (R, \mathbf{m}) be a one-dimensional Noetherian local ring.*

- (1) *If yR is a parameter ideal of R , then $g(Q) \leq g(y)$ for every parameter ideal Q such that $Q \subseteq yR$.*
- (2) *There exists a positive integer n such that all parameter ideals of R contained in \mathbf{m}^n have the same Goto number. Moreover, this number is the minimal Goto number of a parameter ideal of R .*

Proof. If $Q = qR$ is a parameter ideal and $Q \subseteq yR$, then $q = yz$ for some $z \in R$. If $Q = yR$, then $g(Q) = g(y)$, while if Q is properly contained in yR , then zR is a parameter ideal, and Corollary 1.7 implies that $g(Q) \leq g(y)$. This establishes item (1). For item (2), let yR be a parameter ideal such that $g(y)$ is the minimal element of the set

$$\mathcal{G}(R) = \{g(Q) \mid Q \text{ is a parameter ideal of } R\}.$$

Since yR is a parameter ideal, there exists a positive integer n such that $\mathbf{m}^n \subseteq yR$. By item (1), $g(Q) \leq g(y)$ for every parameter ideal $Q \subseteq \mathbf{m}^n$, and by the choice of $g(y)$, we have $g(Q) = g(y)$ is the minimal Goto number of a parameter ideal of R . \square

Remark 3.2. Let $g = g(Q)$ denote the Goto number of the parameter ideal Q . The chain of ideals

$$Q = Q : \mathbf{m}^0 \subsetneq Q : \mathbf{m} \subsetneq Q : \mathbf{m}^2 \subsetneq \cdots \subsetneq Q : \mathbf{m}^g \subseteq \overline{Q}$$

implies that the length $\ell_R(\overline{Q}/Q)$ of the R -module \overline{Q}/Q is an upper bound on $g(Q)$. Thus if (R, \mathbf{m}) is a one-dimensional Noetherian reduced ring¹ such that \overline{R} is a finitely generated R -module, then the length of \overline{R}/R is an upper bound for $g(Q)$ and therefore the set $\mathcal{G}(R)$ is finite. To see this, let $Q = qR$ be a parameter ideal of R . Then $q\overline{R}$ is an integrally closed ideal of \overline{R} , and $\overline{Q} = q\overline{R} \cap R$, cf. [13, Proposition 1.6.1]. Thus we have

$$\ell_R(\overline{R}/R) = \ell_R(q\overline{R}/qR) \geq \ell_R(\overline{Q}/Q).$$

Remark 3.3. For certain parameter ideals Q it is possible to compute the Goto number $g(Q)$ as an index of nilpotency. If $Q = xR$ is a reduction of \mathbf{m} , then \mathbf{m} is the integral closure of Q and

$$g(Q) = \max\{i \mid (Q : \mathbf{m}^i) \neq R\} = \min\{i \mid \mathbf{m}^{i+1} \subseteq Q\}$$

is the index of nilpotency of \mathbf{m} with respect to Q [9, (4.4)]. This is an integer that is less than or equal to the reduction number of \mathbf{m} with respect to Q , with equality holding if the associated graded ring $\text{gr}_{\mathbf{m}}(R)$ is Cohen-Macaulay.

We prove in Theorem 3.4 a sharpening of Theorem 3.1 in the case where \overline{R} is module-finite over R .

Theorem 3.4. *Let (R, \mathbf{m}) be a one-dimensional Noetherian local reduced ring such that \overline{R} is module-finite over R . Let $C = R :_R \overline{R}$ be the conductor of R in \overline{R} , and let $x \in \mathbf{m}$ and $y \in C$ generate parameter ideals. Then for each positive integer n , the Goto number $g(x^n y) = g(xy)$. Thus for all parameter ideals $Q = qR \subseteq xC = x\overline{C}$, we have $g(Q) = g(xy)$. Furthermore, this is the minimal possible Goto number of a parameter ideal in R .*

Proof. By Corollary 1.7, $g(xy) \geq g(x^n y)$. To prove that $g(xy) \leq g(x^n y)$, it suffices to prove for each positive integer i that

$$(1) \quad (xyR : \mathbf{m}^i) \subseteq xy\overline{R} \implies (x^n yR : \mathbf{m}^i) \subseteq x^n y\overline{R}.$$

Assume there exists $w \in R$ with $w\mathbf{m}^i \subseteq x^n yR$ and with $w \notin x^n y\overline{R}$. Notice that $xw \in x^n y\overline{R} \subseteq x^n C \subseteq x^n R$ implies $w \in x^{n-1}R$. Therefore by replacing w if necessary by wx^j for some positive integer j , we may assume that $w \in x^{n-1}R$, so

¹If (R, \mathbf{m}) is a Noetherian local ring that is not equal to its total quotient ring and if \overline{R} is module-finite over R , then R is reduced. For if $x \in \mathbf{m}$ is a regular element and $y \in R$ is nilpotent, then $y/x^n \in \overline{R}$, so $y \in x^n \overline{R}$, for each $n \in \mathbb{N}$. But if \overline{R} is module-finite over R , then \overline{R} is Noetherian and $\bigcap_{n=1}^{\infty} x^n \overline{R} = (0)$, cf. [13, Prop. 1.5.2].

$w = x^{n-1}z$ for some $z \in R$. Thus $x^{n-1}z\mathbf{m}^i \subseteq x^n yR$ implies that $z\mathbf{m}^i \subseteq xyR$, so $z \in xyR : \mathbf{m}^i$. Moreover, $w = x^{n-1}z \notin x^n y\overline{R}$ implies that $z \notin xy\overline{R}$. This establishes the implication displayed in (1). Theorem 3.1 implies that for n sufficiently large $g(x^n y)$ is the minimal Goto number of a parameter ideal of R . This completes the proof of Theorem 3.4. \square

In comparison with Theorem 3.4, we demonstrate in Example 4.6 that the Goto number $g(Q)$ of parameter ideals contained in the conductor need not be constant, even in the case where (R, \mathbf{m}) is a Gorenstein numerical semigroup ring.

Theorem 3.5 establishes conditions on a one-dimensional Noetherian local ring R in order that the set $\{\ell_R(\overline{Q}/Q) \mid Q \text{ is a parameter ideal of } R\}$ is finite.

Theorem 3.5. *Let (R, \mathbf{m}) be a one-dimensional Noetherian local ring, let $(\widehat{R}, \widehat{\mathbf{m}})$ denote the \mathbf{m} -adic completion of R , and let \mathbf{n} denote the nilradical of \widehat{R} . The following statements are equivalent.*

- (1) *The length $\ell_{\widehat{R}}(\mathbf{n})$ is finite.*
- (2) *The set $\{\ell_R(\overline{Q}/Q) \mid Q \text{ is a parameter ideal of } R\}$ is finite.*

Proof. By Remark 2.4, item (2) holds for R if and only if it holds for \widehat{R} . Therefore, to prove (1) \iff (2), we may assume that R is complete.

Assume that $\ell_R(\mathbf{n})$ is finite, and let $R' = R/\mathbf{n}$. If Q is a parameter ideal of R , then $\mathbf{n} \subset \overline{Q}$ and $\ell_R((Q + \mathbf{n})/Q) \leq \ell_R(\mathbf{n})$. Since R' is a reduced complete Noetherian local ring, its integral closure is a finite R' -module. Thus by Remark 3.2, the set $\{\ell_{R'}(\overline{Q}R'/QR') \mid Q \text{ is a parameter ideal of } R'\}$ is bounded by some integer s . It follows that $s + \ell_R(\mathbf{n})$ is an upper bound for $\ell_R(\overline{Q}/Q)$, so the set $\{\ell_R(\overline{Q}/Q) \mid Q \text{ is a parameter ideal of } R\}$ is finite.

Assume that $\ell_R(\mathbf{n})$ is infinite and let $Q_1 = xR$ be a parameter ideal of R . For each positive integer n , let $Q_n = x^n R$. Then $Q_n + \mathbf{n} \subseteq \overline{Q_n}$, and

$$\frac{(Q_n + \mathbf{n})}{Q_n} \cong \frac{\mathbf{n}}{(Q_n \cap \mathbf{n})} = \frac{\mathbf{n}}{Q_n \mathbf{n}}.$$

Hence $\ell_R(\overline{Q_n}/Q_n) \geq \ell_R(\mathbf{n}/x^n \mathbf{n})$. Therefore $\ell_R(\overline{Q_n}/Q_n)$ goes to infinity as n goes to infinity. This completes the proof of Theorem 3.5. \square

Corollary 3.6. *With notation as in Theorem 3.5, if the length $\ell_{\widehat{R}}(\mathbf{n})$ is finite, then the set $\mathcal{G}(R) = \{g(Q) \mid Q \text{ is a parameter ideal of } R\}$ is finite.*

Proof. Apply Theorem 3.5 and Remark 3.2. \square

Remark 3.7. Let (R, \mathbf{m}) be a one-dimensional reduced Cohen-Macaulay local ring, and let \widehat{R} denote the \mathbf{m} -adic completion of R . If the nilradical \mathbf{n} of \widehat{R} is nonzero, then $\ell_{\widehat{R}}(\mathbf{n})$ is infinite. For if xR is a parameter ideal of R , then x is a regular element of \widehat{R} , and hence $\{x^n \mathbf{n}\}_{n=1}^{\infty}$ is a strictly descending chain of ideals of \widehat{R} . It is known that $\mathbf{n} = (0)$ if and only if \overline{R} is module finite over R . There are well-known examples of one-dimensional Noetherian local domains R for which \overline{R} is not module finite over R . For such a ring R , Theorem 3.5 implies that the set $\{\ell_R(\overline{Q}/Q) \mid Q \text{ is a parameter ideal of } R\}$ is not finite.

A specific example of a one-dimensional Noetherian local domain R for which \overline{R} is not module finite over R is given by Nagata [12, (E3.2), page 206] and described in [13, Ex. 4.8, page 89]. Let $A = k^p[[X]][k]$, where k is a field of characteristic $p > 0$ such that $[k : k^p] = \infty$. Then A is a one-dimensional regular local ring. The example of Nagata is

$$R = \frac{A[Y]}{(Y^p - \sum_{i \geq 1} b_i^p X^{ip})},$$

where $\{b_i\}_{i=1}^{\infty}$ are elements of k that are p -independent over k^p .

We prove that the set $\mathcal{G}(R)$ of Goto numbers of parameter ideals of R is infinite. By Remark 2.4, it suffices to prove that the completion \widehat{R} of R has this property. Notice that \widehat{R} is a homomorphic image of a two-dimensional regular local domain: indeed, with $S = k[[X, Y]]$, then $\widehat{R} \cong S/Y^p S$, so $\widehat{R} = k[[x, y]]$, where $y^p = 0$. Corollary 3.9 below implies that $\mathcal{G}(R)$ is infinite.

Theorem 3.8. *Let (R, \mathbf{m}) be a one-dimensional Noetherian local ring. If there exists a nonzero principal ideal yR such that R/yR is one-dimensional and Cohen-Macaulay and $(0) : y$ is contained in the nilradical, then the set $\mathcal{G}(R)$ is infinite.*

Proof. The assumption that R/yR is one-dimensional and Cohen-Macaulay implies that each $P \in \text{Ass } R/yR$ is a minimal prime of R . Let

$$x \in \mathbf{m} \setminus \bigcup_{P \in \text{Ass } R/yR} P.$$

If R has minimal primes other than those in $\text{Ass } R/yR$, choose x also to be in each of these other minimal primes of R . For each positive integer n , let $Q_n := (y + x^n)R$. Notice that Q_n is a parameter ideal of R . Checking integral closure modulo minimal primes, we see that $(y, x^n)R + \mathbf{n} \subseteq \overline{Q_n}$, where \mathbf{n} is the nilradical of R . We prove that $g(Q_n) \geq n$. Let $r \in (Q_n : \mathbf{m}^n)$. Then $r \in (Q_n : x^n)$, so $rx^n = a(y + x^n)$, for some $a \in R$. Hence $(r - a)x^n = ay$, so $r - a \in (yR : x^n)$. Since x^n is regular

on R/yR , we have $r - a = by$, for some $b \in R$. It follows that $x^n by = ay$, so $(x^n b - a)y = 0$ and $x^n b - a \in (0) : y \subseteq \mathfrak{n}$. Therefore $a = x^n b + c$, where $c \in \mathfrak{n}$. Hence $r = bx^n + by + c \in \overline{Q_n}$. We conclude that $g(Q_n) \geq n$, and therefore that $\mathcal{G}(R)$ is infinite. \square

Corollary 3.9. *Let (R, \mathfrak{m}) be a one-dimensional Cohen-Macaulay local ring such that \mathfrak{m} is minimally 2-generated. The following are equivalent:*

- (1) $\mathcal{G}(R)$ is finite.
- (2) The \mathfrak{m} -adic completion \widehat{R} of R is reduced.
- (3) \overline{R} is module-finite over R .

Proof. Assume (1). By Remark 2.4, $\mathcal{G}(\widehat{R})$ is finite. The structure theorem for complete local rings [12, (31.1)] implies that \widehat{R} is a homomorphic image of a complete regular local ring. Since \mathfrak{m} is minimally 2-generated, we obtain $\widehat{R} = S/I$, where S is a 2-dimensional regular local ring. Since \widehat{R} is Cohen-Macaulay and $\dim \widehat{R} = 1$, the ideal I is of the form $I = (p_1^{e_1} \cdots p_t^{e_t})$, where p_1, \dots, p_t are non-associate prime elements and e_1, \dots, e_t are positive integers. If $e_i > 1$ for some i , then Theorem 3.8 applied to $y = p_i$ shows that $\mathcal{G}(\widehat{R})$ is infinite, which is a contradiction. So necessarily all e_i equal 1, which proves (2). The implication (2) \implies (3) follows say from [13, Corollary 4.6.2], and (3) \implies (1) follows from Remark 3.2 and Corollary 3.6. \square

Example 3.10. Let S be a 3-dimensional regular local ring with maximal ideal $(u, v, w)S$. Let $I = (u, w)S \cap (v^2, u - w)S$ and let $R = S/I$. Notice that vR is a nonzero principal ideal such that R/vR is one-dimensional and Cohen-Macaulay and such that $(0) :_R v$ is contained in the nilradical. By Theorem 3.8, $\mathcal{G}(R)$ is infinite.

We record in Proposition 3.11 a general ideal-theoretic condition that implies $\mathcal{G}(R)$ is infinite.

Proposition 3.11. *Let (R, \mathfrak{m}) be a one-dimensional Noetherian local ring, and let x, y be elements of R such that for all n , $y + x^n$ is a parameter. Assume that for all n , $(y) : x^n \subseteq \overline{(y + x^n)}$ and $(x^n) : y \subseteq \overline{(y + x^n)}$. Then $\mathcal{G}(R)$ is infinite.*

Proof. We prove that $g(y + x^n) \geq n$. Let $r \in (y + x^n) : \mathfrak{m}^n$. Then $rx^n = a(y + x^n)$ for some $a \in R$. Then $r - a \in ((y) : x^n) \subseteq \overline{(y + x^n)}$ and $a \in ((x^n) : y) \subseteq \overline{(y + x^n)}$, so that $r \in \overline{(y + x^n)}$. \square

We have demonstrated in Remark 3.7 the existence of one-dimensional Noetherian local domains (R, \mathbf{m}) for which the set $\mathcal{G}(R)$ of Goto numbers of parameter ideals is infinite. A question here that remains open is:

Question 3.12. Let (R, \mathbf{m}) be a one-dimensional Noetherian local ring. If the set $\mathcal{G}(R)$ is finite, does R satisfy the equivalent conditions of Theorem 3.5 ?

Theorem 3.8 implies an affirmative answer to Question 3.12 if R is Cohen-Macaulay and \mathbf{m} is 2-generated.

In Proposition 3.13 we obtain an upper bound on the Goto numbers of parameter ideals contained in the conductor in the case where R is Gorenstein. We thank YiHuang Shen for helpful comments regarding Proposition 3.13.

Proposition 3.13. *Let (R, \mathbf{m}) be a one-dimensional Gorenstein local reduced ring such that \overline{R} is module-finite over R . Let $C = R :_R \overline{R}$ be the conductor of R in \overline{R} , and let $Q = qR$ be a parameter ideal contained in C . Then*

$$g(Q) = \max\{i \mid C \subseteq \mathbf{m}^i + Q\}.$$

Proof. Since $q \in C$, we have $\overline{Q} = q\overline{R} \subseteq C$. Also, $qC = qC\overline{R}$, so $QC = \overline{Q}C$. Hence $C \subseteq (Q : \overline{Q})$. Let $r \in (Q : \overline{Q})$. Then $\frac{r}{q}\overline{Q} \subseteq R$. Let $w \in \overline{R}$. Then $qw \in C\overline{R} \cap Q\overline{R} \subseteq R \cap Q\overline{R} = \overline{Q}$, whence $rw = \frac{r}{q}qw \in \frac{r}{q}\overline{Q} \subseteq R$, so that $r \in C$. This proves that $(Q : \overline{Q}) \subseteq C$ and hence $(Q : \overline{Q}) = C$. Now the proposition follows from Proposition 1.9. \square

Remark 3.14. The conclusion of Proposition 3.13 fails if R is not assumed to be Gorenstein. Let R be a numerical semigroup ring associated to the numerical semigroup generated by 4, 5, 11. The conductor $C = R :_R \overline{R} = x^8\overline{R}$, and $Q = x^{12}R$ is a parameter ideal contained in C . The Goto number $g(x^{12}) = 2$, but we have $\max\{i \mid C \subseteq \mathbf{m}^i + x^{12}R\} = 1$, because $x^{11} \notin \mathbf{m}^2 + x^{12}R$.

4. NUMERICAL SEMIGROUP RINGS

This section provides lower and upper bounds on the Goto numbers of parameter ideals in numerical semigroup rings.

Let (R, \mathbf{m}) be a numerical semigroup ring associated to a rank-one discrete valuation ring V as in (1.3) and let G be the numerical semigroup associated to R . Assume that $R \subsetneq V$, or, equivalently, that G is minimally generated by positive integers a_1, \dots, a_d , with $1 < a_1 < \dots < a_d$ and $\gcd(a_1, \dots, a_d) = 1$. Necessarily $d > 1$, and $\mathbf{m} = (x^{a_1}, \dots, x^{a_d})R$ is minimally generated by x^{a_1}, \dots, x^{a_d} .

Theorem 4.1. *Let f denote the Frobenius number of the numerical semigroup ring R . Then*

$$g(x^{f+a_1+1}) = \min\{g(Q) \mid Q \text{ is a parameter ideal of } R\}.$$

Moreover, for all $e \geq f + a_1 + 1$, we have $g(x^e) = g(x^{f+a_1+1})$.

Proof. The conductor C of R into $\overline{R} = V$ is $C = x^{f+1}V$. Apply Theorem 3.4. \square

The lower bound for e given in Theorem 4.1 is sharp: if $G = \langle 9, 19 \rangle$, then $f = 143$, $a_1 = 9$, $f + a_1 + 1 = 153$, and $g(x^{152}) = 9 > \min\{g(x^{a_i}) : i = 1, \dots, d\} = 8$.

Remark 4.2. Corollary 1.7 implies that, for all $e \geq f + a_1 + 1$, one has

$$(2) \quad g(x^e) \leq \min\{g(x^{a_i}) \mid i = 1, \dots, d\}.$$

We prove equality holds in (2) in the case where $d = 2$ in Theorem 5.10 below. However, YiHuang Shen has pointed out that strict inequality may hold in display (2) if $d \geq 3$. In particular, for the semigroup $\langle 7, 11, 20 \rangle$, one has $g(x^7) = 4$, $g(x^{11}) = 4$ and $g(x^{20}) = 5$, while $g(x^{45}) = 3$. Similar strict inequalities occur for the semigroups $\langle 8, 11, 15 \rangle$, $\langle 9, 14, 17 \rangle$, $\langle 10, 13, 18 \rangle$. Even in the case where $d = 3$ and the numerical semigroup is symmetric, YiHuang Shen has found examples where strict inequality holds in display (2). For the symmetric numerical semigroup $\langle 11, 14, 21 \rangle$, one has $g(x^{11}) = 6$, $g(x^{14}) = 6$ and $g(x^{21}) = 7$, while $g(x^{85}) = 5$.

Proposition 4.3. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to a rank-one discrete valuation ring V as in (1.3) and let G be the value semigroup of R . Then*

$$(3) \quad \sup\{g(x^e) \mid e \in G\} = \max\{g(x^{a_j}) \mid j = 1, \dots, d\}.$$

Proof. Apply Corollary 1.7. \square

We clearly have

$$(4) \quad \sup\{g(x^e) : e \in G\} \leq \sup\{g(Q) \mid Q \text{ a parameter ideal in } R\}.$$

Strict inequality may hold in display (4) as we demonstrate in Example 4.4.

Example 4.4. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle 4, 7, 9 \rangle$. Then $(x^4) : \mathbf{m}^3$ contains 1, $(x^7) : \mathbf{m}^3$ contains x^4 , and $(x^9) : \mathbf{m}^3$ contains x^8 . Therefore display (3) implies that $\sup\{g(x^e) : e \in G\} \leq 2$. However, $(x^7 + x^8 + x^9) : \mathbf{m}^3 = (x^7 + x^8 + x^9, x^7 + x^9 + x^{11}, x^7 - x^{13}, x^7 - x^{16}, x^9 - x^{18})$, so that $g(x^7 + x^8 + x^9) \geq 3$.

We are interested in bounds for the Goto numbers of arbitrary parameter ideals of a numerical semigroup ring. Theorem 4.1 gives a general lower bound. Proposition 4.5 gives a relative lower bound for each parameter ideal in terms of the Goto number of the monomial parameter ideal with the same integral closure.

Proposition 4.5. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to a rank-one discrete valuation ring V as in (1.3) and let G be the value semigroup of R . Let $Q = qR$ be a parameter ideal of R . Then $q = ux^b$, where $b \in G$ and u is a unit of V , and we have $g(Q) \geq g(x^b)$.*

Proof. Let $r = wx^c \in R$, where w is a unit of V and $c \in G$ with $c < b$. It suffices to prove for each positive integer i that $wx^c \mathbf{m}^i \subseteq ux^b R$ implies that $x^c \mathbf{m}^i \subseteq x^b R$. Now \mathbf{m}^i is generated by elements of the form x^a , where $a \in G$. Using part (5) of Remark 1.4, we see that $wx^c x^a \in ux^b R$ implies that $c + a - b \in G$, and this implies that $x^{c+a} \in x^b R$. \square

With notation as in Proposition 4.5, it may happen that $g(Q) > g(x^b)$ even in the case where R is Gorenstein and $b > f$, as we demonstrate in Example 4.6.

Example 4.6. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle 5, 11 \rangle$. Then $f = 39$, and $g(x^{40}) = 4 < g(x^{40} + x^{44}) = 5$. Note that $x^b = x^{40}$ and $ux^b = x^{40} + x^{44}$ are in the conductor C of R in V .

Theorem 4.7 is due to Lance Bryant. It gives an upper bound on the Goto number of parameter ideals.

Theorem 4.7. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to a rank-one discrete valuation ring V as in (1.3). Assume that G is minimally generated by a_1, \dots, a_d , with $1 < a_1 < \dots < a_d$, and let f be the Frobenius number of R . For all parameter ideals Q in R , we have*

$$(5) \quad g(Q) \leq \left\lfloor \frac{f}{a_1} \right\rfloor + 1 = \left\lceil \frac{f}{a_1} \right\rceil.$$

Proof. Let Q be a parameter ideal of R . As observed in Remark 1.4, $Q = ux^c R$, where $c \in G$, and $u = 1 + \sum_{i=1}^f u_i x^i$, where each of the u_i is either zero or a unit of R . Let $m = \left\lfloor \frac{f}{a_1} \right\rfloor + 1$. It suffices to prove that $Q : \mathbf{m}^{m+1}$ contains an element that is not integral over Q . Since Q is a parameter ideal, $c > 0$. Let b be the largest element in G that is strictly smaller than c . Then $c - b \leq a_1$, so $b - c \geq -a_1$. Let

e_i be positive integers such that $\sum e_i = m + 1$. Then

$$b + \sum e_i a_i - c \geq b + \sum e_i a_1 - c \geq (m)a_1 > f.$$

Therefore $x^b \mathbf{m}^{m+1} \subseteq (x^c)C$, where $C = R_{>f}$ is the conductor of R in V . Since $x^c C = (ux^c)u^{-1}C$ and $u^{-1}C \subseteq R$, we have $x^b \mathbf{m}^{m+1} \subseteq Q$, so $x^b \in Q : \mathbf{m}^{m+1}$. Since $b < c$, the element x^b is not integral over $Q = ux^c R$. This completes the proof. \square

Remark 4.8. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, a_2 \rangle$. In Theorem 5.5 below, we prove that the Goto number $g(x^{a_2}) = \left\lfloor \frac{a_2 - b_2 + f}{a_1} \right\rfloor = a_2 - 1 - \left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor$ is a sharp upper bound for the Goto number of monomial parameter ideals of R . Theorem 4.7 implies that

$$\sup\{g(Q) \mid Q \text{ is a parameter ideal of } R\} \leq \left\lfloor \frac{f}{a_1} \right\rfloor + 1.$$

It is well known that if $G = \langle a_1, a_2 \rangle$, then the Frobenius number $f = a_1 a_2 - a_1 - a_2$, cf. [13, Example 12.2.1]. Thus $\left\lfloor \frac{f}{a_1} \right\rfloor + 1 = \left\lfloor \frac{a_1 a_2 - a_1 - a_2}{a_1} \right\rfloor + 1$. Since $\left\lfloor \frac{-a_2}{a_1} \right\rfloor = -\left\lceil \frac{a_2}{a_1} \right\rceil$ and a_1 and a_2 are relatively prime, we see that $g(x^{a_2}) = \left\lfloor \frac{f}{a_1} \right\rfloor + 1$. Therefore, in the case where $d = 2$, the upper bound given in Theorem 4.7 is a sharp upper bound for the Goto numbers of parameter ideals of R , and this upper bound is attained by the monomial parameter ideal (x^{a_2}) .

Remark 4.9. The upper bound $g(Q) \leq \left\lfloor \frac{f}{a_1} \right\rfloor$ given in Theorem 4.7 is not always a sharp upper bound for the Goto numbers of parameter ideals of a numerical semigroup ring. YiHuang Shen has constructed a family of examples that illustrate this, the simplest example being $G = \langle 4, 6, 7 \rangle$. If (R, \mathbf{m}) is a numerical semigroup ring associated to the semigroup $G = \langle 4, 6, 7 \rangle$, then $f = 9$, so 3 is the upper bound given by Theorem 4.7, while $g(Q) = 2$ for each parameter ideal Q of R .

5. NUMERICAL SEMIGROUP RINGS – MONOMIAL IDEALS

As in Section 4, let (R, \mathbf{m}) be a numerical semigroup ring associated to a rank-one discrete valuation ring V , and let $G = \langle a_1, \dots, a_d \rangle$ be the numerical semigroup associated to R . In this section we establish bounds for the Goto numbers of monomial parameter ideals in R . It is well known that numerical semigroups follow varied patterns that are difficult to classify precisely. For example, in the case where $d \geq 4$, there is no known closed formula for the Frobenius number of R in terms of the minimal generators a_1, \dots, a_d of G .

Proposition 5.1. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, \dots, a_d \rangle$, and let f denote the Frobenius number of R . For each $j > 1$, we have*

$$(6) \quad g(x^{a_j}) \leq \left\lfloor \frac{a_j - b_j + f}{a_1} \right\rfloor,$$

where b_j is the largest element of G that is strictly smaller than a_j .

Proof. Set $b = \lfloor \frac{a_j - b_j + f}{a_1} \rfloor$. We prove that $(x^{a_j}) : \mathbf{m}^{b+1}$ contains x^{b_j} . Let $c_i \in \mathbb{N}$ be such that $\sum_{i=1}^d c_i = b + 1$. Then

$$b_j + \sum_{i=1}^d a_i c_i \geq b_j + \sum_{i=1}^d a_i c_i = b_j + a_1(b + 1) > b_j + (a_j - b_j + f) = a_j + f.$$

Since this inequality is strict, $b_j + \sum_{i=1}^d a_i c_i - a_j \in G$. Therefore

$$x^{b_j} (x^{a_1})^{c_1} \dots (x^{a_d})^{c_d} \in x^{a_j} R.$$

This proves that $(x^{a_j}) : \mathbf{m}^{b+1}$ contains x^{b_j} . Since $b_j < a_j$, the element x^{b_j} is not integral over (x^{a_j}) . Thus $g(x^{a_j}) \leq b$. \square

The inequality given in display (6) may be strict as we demonstrate in Example 5.2.

Example 5.2. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle 9, 19, 21 \rangle$. One can compute that the Frobenius number f of R is 71. For $j = 3$, $\lfloor \frac{a_j - b_j + f}{a_1} \rfloor = \lfloor \frac{21 - 19 + 71}{9} \rfloor = 8$, but $g(x^{21}) = 6$. However, for $j = 2$, $\lfloor \frac{a_j - b_j + f}{a_1} \rfloor = \lfloor \frac{19 - 18 + 71}{9} \rfloor = 8$ is indeed $g(x^{19})$.

Proposition 5.3. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, \dots, a_d \rangle$, and let f denote the Frobenius number of R . Then*

$$(7) \quad g(x^{a_1}) \leq \left\lceil \frac{f + a_1 + 1}{a_2} \right\rceil - 1.$$

Proof. It suffices to prove that $1 \in (x^{a_1}) : \mathbf{m}^{\lceil \frac{f + a_1 + 1}{a_2} \rceil}$, and for this it suffices to prove that whenever $c_i \in \mathbb{N}$ and $\sum_i c_i = \lceil \frac{f + a_1 + 1}{a_2} \rceil$, then $\sum_i c_i a_i - a_1 \in G$. In proving this, we may assume that $c_1 = 0$. Then

$$\sum_i c_i a_i - a_1 \geq \sum_i c_i a_2 - a_1 \geq f + a_1 + 1 - a_1 > f.$$

This completes the proof of Proposition 5.3. \square

In Example 5.2, where $G = \langle 9, 19, 21 \rangle$, the inequality in display (7) is an equality since $\lceil \frac{f+a_1+1}{a_2} \rceil - 1 = 4 = g(x^9)$. However, if $G = \langle 5, 6, 13 \rangle$ is the value semigroup of R , then $f = 14$ and $\lceil \frac{f+a_1+1}{a_2} \rceil - 1 = 3 > 2 = g(x^5)$.

Concerning upper bounds for the Goto number of monomial parameter ideals, as observed in Proposition 4.3, we have

$$\rho := \sup\{g(x^e) \mid e \in G\} = \max\{g(x^{a_1}), \dots, g(x^{a_d})\},$$

and Propositions 5.1 and 5.3 imply that

$$(8) \quad \rho \leq \max \left\{ \left\lceil \frac{f+a_1+1}{a_2} \right\rceil - 1, \left\lfloor \frac{a_2-b_2+f}{a_1} \right\rfloor, \dots, \left\lfloor \frac{a_d-b_d+f}{a_1} \right\rfloor \right\},$$

where b_j is the largest element of G that is strictly smaller than a_j , for each j with $2 \leq j \leq d$. The maximum in display (8) is at most $1 + \frac{f}{a_1}$, because

$$b_i \in \{a_i - a_1, a_i - a_1 + 1, \dots, a_i - 1\}.$$

The upper bound given in display (8) for the Goto numbers of monomial parameter ideals may fail to be sharp as we demonstrate in Example 5.4.

Example 5.4. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle 4, 7, 9 \rangle$. Then $\rho = \max\{g(x^4), g(x^7), g(x^9)\} = 2$. However, the Frobenius number $f = 10$ and

$$\max \left\{ \left\lceil \frac{f+a_1+1}{a_2} \right\rceil - 1, \left\lfloor \frac{a_2-b_2+f}{a_1} \right\rfloor, \left\lfloor \frac{a_3-b_3+f}{a_1} \right\rfloor \right\} = 3.$$

Theorem 5.5 shows that the inequalities in Propositions 5.1 and 5.3 are equalities if $d = 2$. We use the well-known fact that if $G = \langle a_1, a_2 \rangle$, then the Frobenius number $f = a_1a_2 - a_1 - a_2$, cf. [13, Example 12.2.1].

Theorem 5.5. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, a_2 \rangle$, then*

$$\begin{aligned} g(x^{a_1}) &= \left\lceil \frac{f+a_1+1}{a_2} \right\rceil - 1 = a_1 - 1 \\ &\leq g(x^{a_2}) = \left\lfloor \frac{a_2-b_2+f}{a_1} \right\rfloor = a_2 - 1 - \left\lfloor \frac{a_2-1}{a_1} \right\rfloor. \end{aligned}$$

Proof. Using that $f = a_1a_2 - a_1 - a_2$, we see that $\left\lceil \frac{f+a_1+1}{a_2} \right\rceil - 1 = a_1 - 1$. Thus Proposition 5.3 implies that $g(x^{a_1}) \leq a_1 - 1$. Since $a_1a_2 - a_1 - a_2 \notin G$, we have $(x^{a_2})^{a_1-1} \notin x^{a_1}R$. Therefore $(x^{a_1}R : \mathbf{m}^{a_1-1}) \subseteq \mathbf{m}$, and thus is integral over $x^{a_1}R$. Hence the Goto number $g(x^{a_1}) = a_1 - 1$.

Let $s = \left\lfloor \frac{a_2 - b_2 + f}{a_1} \right\rfloor$. It is clear that $b_2 = \left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor a_1$. Thus $s = a_2 - 1 - \left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor$. Proposition 5.1 implies that $g(x^{a_2}) \leq s$. If $g(x^{a_2}) < s$, then for some nonnegative integer $e < \frac{a_2}{a_1}$, we have $ea_1 + sa_1 - a_2 \in G$. Since $s = a_2 - 1 - \left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor$, we have

$$ea_1 + sa_1 - a_2 = (e - \left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor)a_1 + f.$$

But $(\left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor - e)a_1 \in G$ implies that $f \in G$, a contradiction. Hence $g(x^{a_2}) = s$.

It remains to prove that $g(x^{a_1}) \leq g(x^{a_2})$. Let $r_i \in [0, a_i - 1] \cap \mathbb{N}$, $1 \leq i \leq 2$, be such that $\left\lfloor \frac{f + a_1 + 1}{a_2} \right\rfloor = \frac{f + a_1 + 1 + r_2}{a_2}$, and such that $\left\lfloor \frac{a_2 - 1}{a_1} \right\rfloor = \frac{a_2 - 1 - r_1}{a_1}$. Then $g(x^{a_1}) \leq g(x^{a_2})$ if and only if $\frac{f + a_1 + 1 + r_2 - a_2}{a_2} \leq \frac{(a_2 - 1)a_1 - (a_2 - 1 - r_1)}{a_1}$, which holds if and only if $r_2 a_1 - r_1 a_2 \leq a_1 a_2^2 + a_1 a_2 - a_2^2 + a_2 - a_1^2 a_2 - a_1$. But $r_2 a_1 - r_1 a_2 \leq r_2 a_1 \leq (a_2 - 1)a_1$, and it suffices to prove that $(a_2 - 1)a_1 \leq a_1 a_2^2 + a_1 a_2 - a_2^2 + a_2 - a_1^2 a_2 - a_1$. By assumption $a_1 + 1 \leq a_2$, so that $a_2(a_1^2 - a) \leq a_2^2(a_1 - 1)$, which expands to the desired inequality. This completes the proof of Theorem 5.5. \square

Now we turn to further characterizations of the eventual stable Goto number of parameter ideals.

Proposition 5.6. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, \dots, a_d \rangle$, and let f denote the Frobenius number of R . Let t be the maximum integer such that for all $\alpha \in \{1, 2, \dots, a_1\}$, $\mathbf{m}^t \not\subseteq x^\alpha R$ (R -module containment). Then $t = g(x^{f+a_1+1})$.*

Proof. Since $d > 1$, we have $\mathbf{m} \not\subseteq x^\alpha R$ for all prescribed α . Thus there exist positive integers k such that $\mathbf{m}^k \not\subseteq x^\alpha R$. There is an upper bound on such k , for if k is such that $(k - 2)a_1 > f$, then $\mathbf{m}^k \subseteq x^\alpha R$. Thus an integer t as in the statement of Proposition 5.6 exists.

If $x^l \in (x^{f+a_1+1}) : \mathbf{m}^t$, then by possibly multiplying by a power of x^{a_1} , we may assume without loss of generality that $l \geq f + 1$. Then $\mathbf{m}^t \subseteq (x^{a_1+f+1-l})R$, so that by the definition of t , $l \geq f + a_1 + 1$. This proves that $t \leq g(x^{f+a_1+1})$.

Also by the definition of t , there exists $\alpha \in \{1, 2, \dots, a_1\}$ such that $\mathbf{m}^{t+1} \subseteq x^\alpha R$. Then $x^{f+a_1+1-\alpha} \cdot \mathbf{m}^{t+1} \subseteq (x^{f+a_1+1})R$. Hence $t + 1 > g(x^{f+a_1+1})$. \square

Proposition 5.7. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, \dots, a_d \rangle$, and let f denote the Frobenius number of R . For each $\alpha \in \{1, \dots, a_1\}$, find elements $\beta \in G$ such that $\beta - \alpha \notin G$. Among all such β , fix one for which x^β has the largest \mathbf{m} -adic order. As α varies, let t' be the smallest of these orders. Then $t' = g(x^{f+a_1+1})$.*

Proof. Observe that the β as in the statement exist: 0 works, or by minimality of the generators, either a_1 or a_2 work for each α . Necessarily $\beta - \alpha \leq f$. For each α , let β_α be an element in G such that $\beta_\alpha - \alpha \notin G$ and such that x^{β_α} has the largest \mathbf{m} -adic order. Let t be as in Proposition 5.6. Note that $t = g(x^{f+a_1+1})$.

Let α be such that the corresponding β_α yields the smallest order, namely t' . By assumption, $\mathbf{m}^t \not\subseteq x^\alpha R$. Thus there exists $\gamma \in G$ such that $x^\gamma \in \mathbf{m}^t$ and $\gamma - \alpha \notin G$. Hence $t' \geq t$.

Now suppose that $t' > t$. Then there exists $\alpha \in \{1, \dots, a_1\}$ such that $\mathbf{m}^{t'} \subseteq x^\alpha R$. Hence the \mathbf{m} -adic order of x^{β_α} is strictly smaller than t' , which is a contradiction. Thus $t' \leq t$. \square

Corollary 5.8. *With notation as in Proposition 5.7, the \mathbf{m} -adic order of the conductor ideal $C = x^{f+1}V$ is less than or equal to the Goto number $g(x^{f+a_1+1})$.*

Proof. For each $\alpha \in \{1, \dots, a_1\}$, the element $f + \alpha$ is in G and has the property that subtracting α gives an element not in G . Hence Proposition 5.7 implies Corollary 5.8. \square

With notation as in Corollary 5.8, the element $f + \alpha$ is the largest element of G having the property that subtracting α gives an element not in G . However, in general, $x^{f+\alpha}$ need not have the largest possible \mathbf{m} -adic order, as we demonstrate in Example 5.9.

Example 5.9. Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle 7, 9, 20 \rangle$. The Frobenius number $f = 33$, and the \mathbf{m} -adic order of $x^{33+7} = x^{40} = x^{f+a_1}$ is 2, whereas $38 \in G$, $38 - 7 \notin G$, and the \mathbf{m} -adic order of x^{38} is 3. In fact, the Goto number $g(x^{f+a_1+1}) = 3$, so the \mathbf{m} -adic order of the conductor ideal $C = x^{f+1}V$ is here strictly smaller than $g(x^{f+a_1+1})$.

Theorem 5.10. *Let (R, \mathbf{m}) be a numerical semigroup ring associated to the semigroup $G = \langle a_1, a_2 \rangle$, then*

$$g(x^{f+a_1+1}) = g(x^{a_1}) = \min\{g(x^{a_i}) : i = 1, 2\}.$$

Proof. The last equality is proved in Theorem 5.5. There it was also proved that $g(x^{a_1}) = a_1 - 1$. By Theorem 3.1, $g(x^{f+a_1+1}) \leq g(x^{a_1})$. By Proposition 5.6, to prove Theorem 5.10, it suffices to prove for all $\alpha \in \{1, \dots, a_1\}$, that $\mathbf{m}^{a_1-1} \not\subseteq x^\alpha R$. Let $r \in \{0, \dots, a_1 - 1\}$ be such that $ra_2 \equiv -\alpha \pmod{a_1}$. Such r exists because a_1 and a_2 are relatively prime. Then $(x^{a_1})^r (x^{a_2})^{a_1-1-r} \in \mathbf{m}^{a_1-1}$. Observe that

$a_1a_2 - a_1 - a_2 - (ra_1 + (a_1 - 1 - r)a_2 - \alpha) = -a_1 + r(a_2 - a_1) + \alpha$ is by construction of r an integer multiple of a_1 . If it were negative, then $-a_1 + r(a_2 - a_1) + \alpha \leq -a_1$, which is a contradiction. So $a_1a_2 - a_1 - a_2 - (ra_1 + (a_1 - 1 - r)a_2 - \alpha)$ is a non-negative multiple of a_1 . Thus that non-negative multiple of a_1 plus $ra_1 + (a_1 - 1 - r)a_2 - \alpha$ equals $a_1a_2 - a_1 - a_2 = f$, which is not in G . Hence $ra_1 + (a_1 - 1 - r)a_2 - \alpha$ is not in G , which proves that $\mathfrak{m}^{a_1-1} \not\subseteq x^\alpha R$. \square

REFERENCES

- [1] W. Bruns and J. Herzog, *Cohen-Macaulay ring*, revised ed., Cambridge University Press, Cambridge 1998.
- [2] A. Corso, C. Huneke and W. Vasconcelos, On the integral closure of ideals, *Manuscripta Math.* **95** (1998), 331-347.
- [3] A. Corso, C. Huneke, D. Katz and W. Vasconcelos, Integral closure of ideals and annihilators of local cohomology, in *Lecture in Pure and Applied Mathematics*, **244**,
- [4] A. Corso and C. Polini, Links of prime ideals and their Rees algebras, *J. Algebra* **178** (1995), 224-238.
- [5] A. Corso, C. Polini and W. Vasconcelos, Links of prime ideals, *Math. Proc. Camb. Phil. Soc.* **115** (1995), 431-436.
- [6] S. Goto, Integral closedness of complete intersection ideals, *J. Algebra* **108** (1987), 151-160.
- [7] S. Goto, N. Matsuoka and R. Takahashi, Quasi-socle ideals in a Gorenstein local ring, [arXiv:math.AC/0612333](https://arxiv.org/abs/math.AC/0612333).
- [8] D. Grayson and M. Stillman, Macaulay2. 1996. A system for computation in algebraic geometry and commutative algebra, available via anonymous `ftp` from `math.uiuc.edu`.
- [9] W. Heinzer, M-K Kim and B. Ulrich, The Gorenstein and complete intersection properties of associated graded rings, *J. Pure Appl. Algebra* **201** (2005), 264-283.
- [10] J. Herzog and E. Kunz, *Die Wertehalbgruppe eines lokalen Rings der Dimension 1*, Heidelberg Akad. Wiss. Math. Springer-Verlag, Berlin, 1971.
- [11] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1989.
- [12] M. Nagata, *Local Rings*, Interscience, New York, 1962.
- [13] I. Swanson and C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.
- [14] H.-J. Wang, Links of symbolic powers of prime ideals, preprint, [arXiv:math.AC/0212162](https://arxiv.org/abs/math.AC/0212162).

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907
E-mail address: `heinzer@math.purdue.edu`

DEPARTMENT OF MATHEMATICS, REED COLLEGE, PORTLAND, OR 97202
E-mail address: `iswanson@reed.edu`